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A RELATION BETWEEN THE LOGARITHMIC DERIVATIVES OF RIEMANN AND SELBERG ZETA FUNCTIONS AND A PROOF OF THE RIEMANN HYPOTHESIS UNDER AN ASSUMPTION ON A DISCRETE SUBGROUP OF $SL(2,\mathrm{R})$

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1. Introduction

Let $\zeta(s)$ be the Riemann's zeta function and $\eta(r)$ $(r=\sqrt{-1}(1/2-s))$ the logarithmic derivative of ζ which is of the form:

$$\eta(r)=\sum_{p\in Prim}\sum_{n\geq 1}(\log p)e^{-n(\log p)s}$$
$$=\sum_{i\geq 1}\sum_{n\geq 1}a_{in}e^{-\sqrt{-1}n(\log p_i)r}, \tag{1}$$

where $Prim=\{p_i; i\geq 1\}$ is the set of prime numbers and $a_{in}=(\log p_i)e^{-n(\log p_i)/2}$. This series converges absolutely and uniformly in any half plane $\Im(r)<-1/2-\varepsilon$ $(\varepsilon>0)$ and has meromorphic continuation to the whole complex plane. Then the Riemann Hypothesis that the roots of $\zeta(s)$ all do lie on $\Re(s)=1/2$ is equivalent to showing that the non imaginaly poles of $\eta(r)$ all do lie on $\Im(r)=0$.

Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup of G and Γ a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then for each character χ of a finite dimensional unitary representation of Γ, Gangolli[**G1**] investigates a zeta function $Z_\Gamma(s,\chi)$ of Selberg's type, Selberg[S] originally introduced into the case of $SL(2,\mathbf{R})$. The logarithmic derivative $\eta_G(r)$ of $Z_\Gamma(s,\chi)$ $(r=\sqrt{-1}(\rho_0-s)$ and ρ_0 is a positive real number depending only on $(G,K))$ is of the form:

$$\eta_G(r)=\kappa\sum_{\delta\in Prim_\Gamma}\sum_{n\geq 1}\sum_{\lambda\in L}u_\delta m_\lambda\chi(\delta^n)\xi_\lambda(h(\delta))^{-n}e^{-nu_\delta s}, \tag{2}$$

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where $Prim_\Gamma$ is a complete set of representatives for the conjugacy classes of prime elements in Γ and u_δ ($\delta \in Prim_\Gamma$) the logarithm of the norm $N(\delta)$ of δ. For other notations refer to [G1]. This series converges absolutely and uniformly in any half plane $\Im(r) < -\rho_0 - \varepsilon$ ($\varepsilon > 0$) and has meromorphic continuation to the whole complex plane. Especially, the poles of $\eta_G(r)$ all do lie on $\Im(r) = 0$ or $\Re(r) = 0$, so the Riemann Hypothesis holds true for $Z_\Gamma(s, \chi)$. In what follows we shall rearrange the series as

$$\eta_G(r) = \sum_{i \geq 1} \sum_{n \geq 1} b_{in} e^{-\sqrt{-1} c_{in} u_{\delta_i} r} \tag{3}$$

for which the exponents satisfy $c_{in} u_{\delta_i} = c_{jm} u_{\delta_j}$ if and only if $i = j$ and $n = m$.

We here note that (1) and (3) are quite similar in their forms. Therefore, if two distributions of $Prim$ and $Prim_\Gamma$ are similar in the logarithm of their norms, it is hoped that η and η_G have the same properties, especially, the Riemann Hypothesis holds for η and then, for ζ also. In this paper we let $G = SL(2, \mathbf{R})$ and make an assumption of magnitude and distance of $N(\delta)$ for $\delta \in Prim_\Gamma$, which guarntees the similarity between the distributions (see **(A)** in §2 and **(B)** in §6). Then, under a week assumption **(A)** we shall obatin an integral expression of η in terms of η_G such as

$$\eta(\nu) = \int_{\mathbf{R} - \sqrt{-1} y} \eta_G(x) H(\nu, x) dx \tag{4}$$

($y = 1/2 + \varepsilon$ and see Proposition 3.3). Unfortunately, this formula is valid only for $\Im(\nu) \leq -L$ (L is a large positive number and see Proposition 5.1). Then, the Riemann Hypothesis is equivalent to showing that the right hand side of (4) has analytic continuation to $\Im(\nu) < 0$ except $\nu = -\sqrt{-1}/2$. Under a strong assumption **(B)** we shall obtain the continuation and prove the Riemann Hypothesis (see Theorem 6.1).

Since $\eta(r)$ and $\eta_G(r)$ have a different growth order as $r \to \infty$ (cf. [E], Chap.9 and [H], Chap.6), we see that the distribution of $Prim$ and the one of norms of $Prim_\Gamma$ does not coincide. On the other hand we know that the prime number theorem that gives an approximation of the number of primes less than a given magnitude holds in an exactly same form for both $Prim$ and $Prim_\Gamma$ (cf. [E],Chap.4 and [H], Chap.2). Therefore, according to these facts we can believe that two distributions of $Prim$ and $Prim_\Gamma$ are similar in their norms. Actually, our strong assumption **(B)** expresses a similarity in the following fashion: there exists an injective map

$$\omega : Prim \quad \to \quad Prim_\Gamma \tag{5}$$

for which $\log N(\omega(p)) \leq 1/4 \log p$ or $\log N(\omega(p)) \leq \log p$ and the distance $\delta(p)$ between $\log N(\omega(p))$ and the nearest element being of the form $\log N(\omega(q))$ ($q \in Prim$) is bounded below by $\sigma(\log N(\omega(p)))^{-\theta}$ for positive constants σ and θ, roughly speaking, $\log N(\omega(p)) \leq \log p$ for almost all $p \in Prim$, but, if $\delta(p)$ is sufficiently small like in the case of twin prime elements, it must be $\log N(\omega(p)) \leq 1/4 \log p$. At present we have no idea to find a discrete subgroup Γ of $SL(2, \mathbf{R})$ satisfying this property, however, we have enough reason to believe that a similarity between $Prim$ and $Prim_\Gamma$ deduces the Riemann Hypothesis.

2. Notations

Let $G = SL(2,\mathbf{R})$ and let χ be the trivial character of Γ. Then $\rho_0 = 1/2$ and the explicit form of η_G is given by

$$\eta_G(r) = \sum_{i\geq 1}\sum_{n\geq 1} \frac{u_i/2}{\sinh(nu_i/2)} e^{-\sqrt{-1}nu_i r}, \tag{6}$$

where $u_i = u_{\delta_i}$, and in (3) $c_{in} = n$ and

$$b_{in}{}^{-1} = 2u_i^{-1}\sinh(nu_i/2) \leq ce^{nu_i/2}. \tag{7}$$

For general references to the basic properties of η_G see [G1], [H] and [S]. We denote the increasing sequence of prime numbers as $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ and the one of the norms of elements in $Prim_\Gamma$ as $N(\delta_1), N(\delta_2), N(\delta_3), \ldots$ respectively. We define $u_i = \log N(\delta_i)$ and

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j)\in\mathbf{N}^2 \\ (m,j)\neq(n,i)}} |nu_i - mu_j| \tag{8}$$

for $i \geq 1$ and $n \geq 1$. Then, each δ_{in} is positive, because $\{u_i; i \geq 1\}$ does not have a finite point of accumulation (see [G2], p.415). Moreover, it is easy to see that there exists a positive constant C such that for each $\alpha \geq 0$ and $\beta \geq 1$

$$\varepsilon_{in} = \varepsilon_{in}(\alpha,\beta,C) = Ce^{-\alpha n(\log p_i)}e^{-\beta nu_i} \leq \delta_{in} \tag{9}$$

for all i and $n \geq 1$. We fix such a pair of α and β till the end of §4.

As said in §1, the Riemann Hypothesis holds for η_G. Actually, the poles of η_G are all simple and are as

$$\{\nu_j; j \in \mathbf{Z}\} \cup \{r_j; 1 \leq j \leq 2M\}, \tag{10}$$

where $\nu_j \in \mathbf{R}$ and $r_j \in \sqrt{-1}\mathbf{R}$ (cf. [G1], Proposition 2.7 and [H], p.68). Then it is known that $\nu_{-j} = -\nu_j$ and the poles of η_G which concentrate along $[-\sqrt{-1}/2, \sqrt{-1}/2]$ can be denoted as

$$\{\nu_0, r_j, \bar{r}_j; 1 \leq j \leq M\}, \tag{11}$$

where we let $r_1, r_2, \ldots, r_M$ be the poles of η_G which concentrate along $[-\sqrt{-1}/2, 0)$ and $\bar{r}_j = -r_j = r_{j+M}$. We denote the residues of η_G at ν_j and r_j by n_j and m_j respectively. Then, $n_{-j} = n_j$ and $m_j = m_{j+M} = 1$ for $1 \leq j \leq M$ (cf. [H], Chap.2).

We fix sufficiently small (resp. large) positive numbers ε and δ (resp. E), and a positive number y such that $1/2 < y \leq 1/2 + \varepsilon$.

3. Transition from η_G to η

Let ϕ be a C^∞ compactly supported function on $\mathbf{R}$ satisfying

$$\begin{aligned} &(i) \quad supp(\phi) \subset (-1,1), \\ &(ii) \quad \phi(0) = 1, \\ &(iii) \quad \phi^{(k)}(0) = 0 \quad (1 \le k \le 2M) \end{aligned} \tag{12}$$

and let

$$h_{in}(t) = \frac{a_{in}}{b_{in}} \phi(\frac{t - n(\log p_i)}{\varepsilon_{in}}) \quad (t \in \mathbf{R}) \tag{13}$$

for $i \ge 1$ and $n \ge 1$. Then it is easy to see that h_{in} satisfies the following conditions.

$$\begin{aligned} &(i) \quad supp(h_{in}) \subset (n(\log p_i) - \varepsilon_{in}, n(\log p_i) + \varepsilon_{in}), \\ &(ii) \quad h_{in}(n(\log p_i)) = \frac{a_{in}}{b_{in}}, \\ &(iii) \quad h_{in}^{(k)}(n(\log p_i)) = 0 \quad (1 \le k \le 2M). \end{aligned} \tag{14}$$

Without loss of generality we may assume that $\varepsilon_{11} \le 1/2 \log 2$ and thus, $supp(h_{in}) \subset [1/2 \log 2, \infty)$ for all i and $n \ge 1$. Here we put $\hat{h}_{in}(x) = (2\pi)^{-1} \int_{\mathbf{R}} h_{in}(z) e^{-\sqrt{-1}xz} dz$ and

$$H(\nu, x) = \sum_{i,n \ge 1} e^{\sqrt{-1}(nu_i - n(\log p_i))x} \hat{h}_{in}(\nu - x) \tag{15a}$$

$$= \sum_{i,n \ge 1} e^{-\sqrt{-1}(n(\log p_i)\nu - nu_i x)} \frac{a_{in}}{b_{in}} \varepsilon_{in} \hat{\phi}(\varepsilon_{in}(\nu - x)). \tag{15b}$$

We now consider a condition for which the series (15) converges. For $\theta \ge 0$ and $1 \le p, q \le \infty$ such that $1/p + 1/q = 1$ we suppose that ν and x satisfy

$$\begin{aligned} &(a_E) \quad -E \le \Im(\nu), \Im(x) \le E, \\ &(b_\theta^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha \le -1/p - \delta \\ -\Im(x) + 1/2 - (1-\theta)\beta \le -1/q - \delta, \end{cases} \end{aligned}$$

where δ is a fixed sufficiently small positive number (see §2). Then, substituting the definition of a_{in} and b_{in} (see (1) and (7)) for (15b), we see that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

$$c \sum_{i,n \ge 1} \log p_i e^{(\Im(\nu) - 1/2)n(\log p_i)} e^{(-\Im_A(x) + 1/2)nu_i} \varepsilon_{in}^{1-\theta} |(\varepsilon_{in}(\nu - x))^\theta \hat{\phi}(\varepsilon_{in}(\nu - x))|. \tag{16}$$

Since $\hat{\phi}$ is rapidly decreasing and is holomorphic of exponential type ≤ 1 (cf. [Su], p.146), for each $N \in \mathbf{N}$ there exists $C_N > 0$ for which

$$|\hat{\phi}(x)| \leq C_N(1+|x|)^{-N} e^{|\Im(x)|} \quad (x \in \mathbf{C}). \tag{17}$$

Therefore, it follows from (9) and (a_E) that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

$$cC^{1-\theta} C_{[\theta]+1} e^{2EC} \sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2-(1-\theta)\alpha)n(\log p_i)} e^{(-\Im(x)+1/2-(1-\theta)\beta)nu_i}, \tag{18}$$

where $[\theta]$ is the greatest integer not exceeding θ. Then, this series converges absolutely and uniformly by $(b_\theta^{p,q})$ and the Hölder's inequality.

Lemma 3.1. *If ν and x satisfy (a_E) and $(b_0^{p,q})$, then the series $H(\nu, x)$ converges absolutely and uniformly, and is holomorphic of ν and x. Moreover, if $(b_\theta^{p,q})(\theta \geq 0)$ is satisfied, there exists a positive constant C such that*

$$|H(\nu, x)| \leq C|\nu - x|^{-\theta}.$$

Throughout this paper we assume the following condition:

(A) There exists a positive constant A such that

$$u_i \leq A \log p_i \quad \text{for all } i \geq 1.$$

Then we can replace $(b_\theta^{p,q})$ with

$$(b_{\theta,\gamma}^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha + \gamma \leq -1/p - \delta, \\ -\Im(x) + 1/2 - (1-\theta)\beta - \gamma/A \leq -1/q - \delta, \end{cases}$$

where $\gamma \geq 0$. We fix such a γ.

We next let $-y \leq -y_0 \leq E$ and

$$(c_{\theta,\gamma,y_0}^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha + \gamma \leq -1/p - \delta, \\ y_0 + 1/2 - (1-\theta)\beta - \gamma/A \leq -1/q - \delta. \end{cases}$$

Then, if ν satisfies (a_E) and $(c_{\theta+1,\gamma,y_0}^{p,q})(\theta \in \mathbf{N})$, it follows similarly as above that

$$\int_{\mathbf{R}-\sqrt{-1}y_0} |x|^\theta |H(\nu, x)| dx$$

$$\leq c \sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2)n(\log p_i)} e^{(y_0+1/2)nu_i} \varepsilon_{in}^{-\theta} \left[\varepsilon_{in} \int_{\mathbf{R}-\sqrt{-1}y_0} |(\varepsilon_{in}x)^\theta \hat{\phi}(\varepsilon_{in}(\nu - x))| dx \right]$$

and by letting $x = (x - \nu) + \nu$,

$$\leq cC^{-\theta} C_{\theta+2} e^{2EC} P_\theta(|\nu|) \sum_{i,n\geq 1} \log p_i e^{(\Im(\nu)-1/2+\theta\alpha+\gamma)n(\log p_i)} e^{(y_0+1/2+\theta\beta-\gamma/A)nu_i}, \tag{19}$$

where P_θ is a polynomial of degree θ with coefficients depending only on θ. Then this series converges absolutely and uniformly by $(c_{\theta+1,\gamma,y_0}^{p,q})$ and the Hölder's inequality.

Lemma 3.2. *Let ν be in a compact set S in the tube domain defined by (a_E) and $(c^{p,q}_{\theta+1,\gamma,y_0})$ ($\theta \in \mathbf{N}$ and $-y \le -y_0 \le E$). Let f be a function on $\mathbf{R} - \sqrt{-1}y_0$ such that $f(x) = O(|x|^\theta)$. Then, there exists a positive constant C for which $\int_{\mathbf{R}-\sqrt{-1}y_0} |f(x)H(\nu,x)|dx \le C$. Especially,*

$$T_{y_0}f(\nu) = \int_{\mathbf{R}-\sqrt{-1}y_0} f(x)H(\nu,x)dx$$

is well-defined and is holomorphic of ν satisfying (a_E) and $(c^{p,q}_{\theta+1,\gamma,y_0})$.

Proposition 3.3. *Let P be a polynomial of degree $k (0 \le k \le 2M)$ and ν satisfy (a_E) and $(c^{p,q}_{k+1,\gamma,y})$. Then,*

$$\begin{aligned} (i) \quad P(\nu)\eta(\nu) &= T_y(P\eta_G)(\nu) \\ &= \int_{\mathbf{R}-\sqrt{-1}y} P(x)\eta_G(x)H(\nu,x)dx, \\ (ii) \qquad\qquad 0 &= \int_{\mathbf{R}-\sqrt{-1}y} P(x)\eta_G(x)H(\nu,-x)dx. \end{aligned}$$

Proof. Since $\eta_G(x) = O(1)$ for $x \in \mathbf{R} - \sqrt{-1}y$ (see [H], Proposition 6.7) and $(c^{p,q}_{k+1,\gamma,y})$ implies $(c^{p,q}_{k+1,\gamma,-y})$, the right hand sides of (i) and (ii) are well-defined and are holomorphic of ν satisfying (a_E) and $(c^{p,q}_{k+1,\gamma,y})$ (see Lemma 3.2). Therefore, we may suppose that $\Im(\nu) \le -y$. Since $mu_j > 0$ for all $m, j \ge 1$, it follows that

$$\begin{aligned} &\int_{\mathbf{R}-\sqrt{-1}y} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx \\ &= \int_{\mathbf{R}} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx. \end{aligned}$$

Then, substituting the definition of $H(\nu,x)$ (see (15a)), we see formally that

$$\begin{aligned} &= \sum_{k,l\ge 1} \int_{\mathbf{R}} e^{-\sqrt{-1}mu_j x} e^{\sqrt{-1}(lu_k - l(\log p_k))x} \hat{h}_{kl}(\nu - x)dx \\ &= \sum_{k,l\ge 1} e^{-\sqrt{-1}(mu_j - lu_k + l(\log p_k))\nu} \int_{\mathbf{R}} e^{\sqrt{-1}(mu_j - lu_k + l(\log p_k))x} \hat{h}_{kl}(x)dx \\ &= \sum_{k,l\ge 1} e^{-\sqrt{-1}(mu_j - lu_k + l(\log p_k))\nu} h_{kl}(mu_j - lu_k + l(\log p_k)). \end{aligned}$$

Since each support of h_{kl} is disjointed from the others, it is easy to see that the condition that $\Im(\nu) \le -y$ guarantees the validity of the above calculation. Moreover, since the support of h_{kl} is contained in $(l(\log p_k) - \varepsilon_{kl}, l(\log p_k) + \varepsilon_{kl})$ and $h_{kl}(l(\log p_k)) = a_{kl} b_{kl}^{-1}$ (see (14)(i) and (ii)), it follows from (9) and the definition of δ_{kl} (see (7)) that

$$
\begin{aligned}
&= \epsilon_{kj}\epsilon_{lm} h_{kl}(l(\log p_k)) e^{-\sqrt{-1}l(\log p_k)\nu} \\
&= \epsilon_{kj}\epsilon_{lm} a_{kl} b_{kl}^{-1} e^{-\sqrt{-1}l(\log p_k)\nu},
\end{aligned}
$$

where $\epsilon_{ij} = 1$ if $i = j$ and 0 otherwise. Therefore, we can deduce that

$$
\begin{aligned}
T_y \eta_G(\nu) &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x) H(\nu, x) dx \\
&= \sum_{j,m\geq 1} b_{jm} \int_{\mathbf{R}-\sqrt{-1}y} e^{-\sqrt{-1}mu_j x} H(\nu, x) dx \\
&= \sum_{j,m\geq 1} a_{jm} e^{-\sqrt{-1}m(\log p_j)\nu} \\
&= \eta(\nu).
\end{aligned} \tag{20}
$$

Here we rewrite $P(\nu)$ as

$$P(\nu) = R_\nu(\nu - x) + P(x),$$

where R_ν is a polynomial of degree k with coefficients depending only on k and ν. Then the formula (i) follows from (20) provided that

$$\int_{\mathbf{R}-\sqrt{-1}y} (\nu - x)^l \eta_G(x) H(\nu, x) dx = 0 \quad (1 \leq l \leq k). \tag{21}$$

We now show (21). If we define $H^{(l)}(\nu, x)$ by replacing h_{in} in (15a) with $(\sqrt{-1})^{-l} h_{in}^{(l)}$, we easily see that the left hand side of (21) is equal to

$$\int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x) H^{(l)}(\nu, x) dx.$$

Obviously, this integral is finite by the condition $(c_{k+1,\gamma,y}^{p,q})$. Then, applying the same argument that deduces (20), especially, by using (14)(iii) instead of (14)(ii), we can show that this integral is equal to 0. The formula (ii) follows by the quite same way. □

We now let ε and δ (resp. E) sufficiently small (resp. large). Then, we can deduce the following,

Corollary 3.4. *The equations* (i) *and* (ii) *in Proposition 3.3 hold for* ν *satisfying*

$$
\begin{cases}
\Im(\nu) - 1/2 + k\alpha + \gamma < -1/p \\
1 + k\beta - \gamma/A < -1/q,
\end{cases}
$$

where $\gamma \geq 0$, $1 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1$.

4. A relation between η and the poles of η_G

We keep the notations and the assumption (A). We first recall that η_G satisfies the functional equation:

$$\eta_G(x) + \eta_G(-x) = cx \tanh \pi x \tag{22}$$

(see [H], Proposition 4.26). In this section we shall express η as the sum of an integral of $x \tanh \pi x$ and the residues of η_G.

Lemma 4.1. *Let P be a polynomial of degree $k (0 \leq k \leq 2M)$ and let ν be in a compact set S satisfying $\Im(S) < 0$, (a_E) and $(c^{p,q}_{k+6,\gamma,0})$. Then the series $\sum_{j \in \mathbf{Z}} n_j P(\nu_j)\, H(\nu, \nu_j)$ converges absolutely and uniformly. Especially, $\sum_{j \in \mathbf{Z}} n_j P(\nu_j)\, H(\nu, \nu_j)$ is well-defined and is holomorphic of ν satisfying $\Im(S) < 0$, (a_E) and $(c^{p,q}_{k+6,\gamma,0})$.*

Proof. Since $\nu_j \in \mathbf{R}$ and $\nu \in S$, Lemma 3.1 implies that for $x \in \mathbf{R}$

$$|H(\nu, x)| \leq C|\nu - x|^{-(k+6)} \sim (1 + |x|)^{-(k+6)}.$$

Then, noting the fact that

$$\sum_{\{j; \nu_j^2 \leq x\}} n_j \quad \sim \quad x^2 \quad (x \to \infty)$$

(see §2 and [G1], Proposition1.2), we see that

$$\begin{aligned} &\sum_{j \in \mathbf{Z}} n_j |P(\nu_j) H(\nu, \nu_j)| \\ \sim &\sum_{j \in \mathbf{Z}} n_j (1 + |\nu_j|)^{-6} \\ \sim &\sum_{k=0}^{\infty} \sum_{k \leq |\nu_j| < k+1} n_j (1 + |\nu_j|)^{-6} \\ \sim &\sum_{k=0}^{\infty} (1 + k)^{-2} < \infty. \qquad \square \end{aligned}$$

We now suppose that ν satisfies $\Im(\nu) < 0$, (a_E) and $(c^{p,q}_{6,\gamma,y})$. We note that, if $|\Im(x)| \leq \varepsilon$, then $x \tanh \pi x = O(|x|)$ and $\eta_G(x) = O(|x|)$ (see [H], Proposition 6.7). Therefore, since $(c^{p,q}_{6,\gamma,y})$ implies $(c^{p,q}_{2,\gamma,\pm\varepsilon})$ and $(c^{p,q}_{6,\gamma,0})$, it follows from Lemma 3.2 and Lemma 4.1 that

$$\begin{aligned} &\int_{\mathbf{R}} cx \tanh \pi x H(\nu, x) dx \\ = &\int_{\mathbf{R}+\sqrt{-1}\varepsilon} cx \tanh \pi x H(\nu, -x) dx \\ = &\int_{\mathbf{R}+\sqrt{-1}\varepsilon} (\eta_G(x) + \eta_G(-x)) H(\nu, -x) dx \\ = &\int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G(x) H(\nu, x) dx + \int_{\mathbf{R}+\sqrt{-1}\varepsilon} \eta_G(x) H(\nu, -x) dx. \end{aligned}$$

The second term is equal to

$$\int_{\mathbf{R}-\sqrt{-1}y}\eta_G(x)H(\nu,-x)dx-\sum_{j\in\mathbf{Z}}n_jH(\nu,\nu_j)-\sum_{1\leq j\leq M}H(\nu,-r_j)$$
$$=-\sum_{j\in\mathbf{Z}}n_jH(\nu,\nu_j)-\sum_{1\leq j\leq M}H(\nu,-r_j)$$

by Proposition 3.3(ii). Therefore, it follows from Proposition 3.3 (i) that

$$\begin{aligned}\eta(\nu)&=\int_{\mathbf{R}-\sqrt{-1}y}\eta_G(x)H(\nu,x)dx\\&=\int_{\mathbf{R}-\sqrt{-1}\varepsilon}\eta_G(x)H(\nu,x)dx+\sum_{1\leq j\leq M}H(\nu,r_j)\\&=\int_{\mathbf{R}}cx\tanh\pi xH(\nu,x)dx+\sum_{j\in\mathbf{Z}}n_jH(\nu,\nu_j)+\sum_{1\leq j\leq 2M}H(\nu,r_j).\end{aligned}$$

Then, letting ε and δ (resp. E) sufficiently small (resp. large), we can obtain the following,

Proposition 4.2. *If ν satisfies*

$$\begin{cases}\Im(\nu)<\min(0,1/2-5\alpha-\gamma-1/p)\\1+5\beta<\gamma/A-1/q,\end{cases}$$

where $\gamma\geq 0$, $1\leq p,q\leq\infty$ and $1/p+1/q=1$, then

$$\eta(\nu)=c\int_{\mathbf{R}}x\tanh\pi xH(\nu,x)dx+\sum_{j\in\mathbf{Z}}n_jH(\nu,\nu_j)+\sum_{1\leq j\leq 2M}H(\nu,r_j).$$

We put

$$P_G(x)=(\nu^2-r_1^2)(\nu^2-r_2^2)\dots(\nu^2-r_M^2). \tag{23}$$

Then, replacing η_G with $P_G\eta_G$, we can obtain the following proposition by the quite same way.

Proposition 4.3. *If ν satisfies*

$$\begin{cases}\Im(\nu)<\min(0,1/2-(5+2M)\alpha-\gamma-1/p)\\1+(5+2M)\beta<\gamma/A-1/q,\end{cases}$$

where $\gamma\geq 0$, $1\leq p,q\leq\infty$ and $1/p+1/q=1$, then

$$\begin{aligned}P_G(\nu)\eta(\nu)&=\int_{\mathbf{R}-\sqrt{-1}\varepsilon}\eta_G(x)P_G(x)H(\nu,x)dx\\&=c\int_{\mathbf{R}}x\tanh\pi xP_G(x)H(\nu,x)dx+\sum_{j\in\mathbf{Z}}n_jP_G(\nu_j)H(\nu,\nu_j).\end{aligned}$$

5. Some modifications

5.1. In the proof of Proposition 3.3 each term $b_{in}e^{-\sqrt{-1}nu_i r}$ of $\eta_G(r)$ $(u_i = \log N(\delta_i))$ transfers to $a_{in}e^{-\sqrt{-1}n(\log p_i)r}$ of $\eta(r)$ under the integral formula. Obviously, to verify such an integral formula δ_i's need not be taken over all elements in $Prim_\Gamma$, and it is enough for each p_i to correspond to a unique element $\delta_{\omega(i)}$ in $Prim_\Gamma$. Actually, for an injective map

$$\omega : \mathbf{N} \quad \rightarrow \quad \mathbf{N}$$

we put

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j)\in\mathbf{N}^2 \\ (m,\omega(j))\neq(n,\omega(i))}} |nu_{\omega(i)} - mu_{\omega(j)}|, \tag{24}$$

$$\varepsilon^{\omega}_{in} = \varepsilon^{\omega}_{in}(\alpha,\beta,C) = Ce^{-\alpha n(\log p_i)}e^{-\beta n u_{\omega(i)}}, \tag{25}$$

$$h^{\omega}_{in} = \frac{a_{in}}{b_{\omega(i)n}}\phi(\frac{t-n(\log p_i)}{\varepsilon^{\omega}_{in}}) \quad (t\in\mathbf{R}), \tag{26}$$

$$H_\omega(\nu,x) = \sum_{i,n\geq 1} e^{\sqrt{-1}(nu_{\omega(i)}-n(\log p_i))x}\hat{h}^{\omega}_{in}(\nu - x) \tag{27}$$

(cf. (8), (9), (13) and (15)). Then it is easy to see that all results in the preceding sections are also valid when we replace $\delta_{in}, \varepsilon_{in}, h_{in}$ and $H(\nu,x)$ by $\delta^{\omega}_{in}, \varepsilon^{\omega}_{in}, h^{\omega}_{in}$ and $H_\omega(\nu,x)$ respectively and **(A)** by

(A)$_\omega$ There exists a positive constant A such that

$$u_{\omega(i)} \leq A\log p_i \quad \text{for all } i\geq 1.$$

5.2. We next modify the η functions. Let

$$\eta^\circ(r) = \sum_{i\geq 1} a_i e^{-\sqrt{-1}(\log p_i)r}, \tag{28}$$

where $a_i = (\log p_i)e^{-(\log p_i)/2}$, and let

$$\eta^\circ_G(r) = \sum_{i\geq 1} b_i e^{-\sqrt{-1}u_i r}, \tag{29}$$

where $b_i = u_i/2\sinh(u_i/2)$. Then, it is easy to see that $\eta(r) - \eta^\circ(r)$ and $\eta_G(r) - \eta^\circ_G(r)$ are holomorphic on $\Im(r) < 0$ (cf. [H], Proposition 3.5). Therefore, in order to prove the Riemann Hypothesis for η it is enough to prove it for η°. Since η° and η°_G inherit all singuralities from η and η_G respectively, the whole arguments in the previous sections except one using the functional equation (22) are also applicable to η° and η°_G. Especially, if we define δ^ω_i, $\varepsilon^\omega_i(\alpha,\beta,C)$, h^ω_i and $H^\circ_\omega(\nu,x)$ by eliminating the sufix n in (24)-(27) respectively, we see that all the results in §2 and §3 are also valid when we replace η, η_G and H by η°, η°_G and H°_ω respectively and **(A)** by **(A)**$_\omega$.

5.3. We now let

$$\omega : D \quad\to\quad \mathbf{N}, \quad D \subset \mathbf{N}$$

be an injective map, and for each $i \in D$ we define $\delta_i^\omega, \varepsilon_i^\omega(\alpha, \beta, C)$ and h_i^ω as above. Moreover, we put

$$\eta_\omega^\circ(r) = \sum_{i \in D} a_i e^{-\sqrt{-1}(\log p_i) r}, \tag{30}$$

$$H_\omega^\circ(\nu, x) = \sum_{i \in D} e^{\sqrt{-1}(n u_{\omega(i)} - n(\log p_i))x} \hat{h}_{in}^\omega(\nu - x) \tag{31}$$

and we define the corresponding assumption $(\mathbf{A})_\omega$, we denote by the same letter, by replacing $i \geq 1$ with $i \in D$. Then repeating the same arguments in §3, especially, taking γ sufficiently large in Corollary 3.4 and Proposition 4.3, we can deduce that

Proposition 5.1. *Let us suppose that* $(\mathbf{A})_\omega$ *holds. Then there exists a positive constant* L *such that if* $\Im(\nu) \leq -L$,

$$(i) \quad \eta_\omega^\circ(\nu) = \int_{\mathbf{R} - \sqrt{-1}y} \eta_G^\circ(x) H_\omega^\circ(\nu, x) dx,$$

$$(ii) \quad P_G(\nu) \eta_\omega^\circ(\nu) = \int_{\mathbf{R} - \sqrt{-1}\varepsilon} P_G(x) \eta_G(x) H_\omega^\circ(\nu, x) dx.$$

6. A proof of the Riemann Hypothesis under an assumption

We retain the notations in the previous sections. We here make an assumption on magnitude and distance of $u_i (i \in \mathbf{N})$, which is stronger than $(\mathbf{A})$, and then give a proof of the Riemann Hypothesis. The assumption can be stated as follows.

$(\mathbf{B})$ There exist an injective map $\omega : \mathbf{N} \to \mathbf{N}$ and positive constants σ and θ for which, except a finite number of i, one of the following conditions holds:

$$(B1) \quad u_{\omega(i)} \leq 1/4 \log p_i,$$

$$(B2) \quad u_{\omega(i)} \leq \log p_i \quad \text{and} \quad \sigma u_{\omega(i)}^{-\theta} \leq \delta_i^\omega.$$

We here put $D_\ell = \{i \in \mathbf{N};\ (B\ell) \text{ holds}\}$ for $\ell = 1, 2$ and $D_3 = \mathbf{N} - D_1 \cup D_2$. In what follows for each $\omega_\ell = \omega|_{D_\ell} (\ell = 1, 2, 3)$ we shall prove that $P_G(\nu) \eta_{\omega_\ell}^\circ(\nu)$ $(\ell = 1, 2, 3)$ (see (30)) is holomorphic on $-2L \leq \Im(\nu) \leq -3\varepsilon$.

$\eta_{\omega_1}^\circ$: Since $(B1)$ implies $(\mathbf{A})_{\omega_1}$ (see 5.3), it follows from Proposition 5.1 that

$$\eta_{\omega_1}^\circ(\nu) = \int_{\mathbf{R} - \sqrt{-1}y} \eta_G^\circ(x) H_{\omega_1}^\circ(\nu, x) dx, \tag{32}$$

if $\Im(\nu) \leq -L$. We now recall the definition of $\varepsilon_i^{\omega_1}$ (see 5.3 and (9)). Then, we can choose a sufficiently small positive number τ depending on ε such that

$$\sum_{i \in D_1} e^{-(1+3\varepsilon) u_{\omega_1(i)}} (\varepsilon_i^{\omega_1})^{-\tau} < \infty. \tag{33}$$

Then, by $(B1)$ and the argument used in (16)-(18) we see that if $-2L \le \Im(\nu) \le -2\varepsilon$ and $\Im(x) = -y = -1/2 - \varepsilon$,

$$
\begin{aligned}
|H^\circ_{\omega_1}(\nu, x)| &\le c \sum_{i \in D_1} \log p_i e^{(-2\varepsilon - 1/2)\log p_i} e^{(\varepsilon+1)u_{\omega_1(i)}} (\varepsilon_i^{\omega_1})^{-\tau} |\nu - x|^{-(1+\tau)} \\
&\le c|\nu - x|^{-(1+\tau)} \sum_{i \in D_1} e^{-(1+3\varepsilon)u_{\omega_1(i)}} (\varepsilon_i^{\omega_1})^{-\tau} \\
&\le c|\nu - x|^{-(1+\tau)} \qquad \text{by (33).}
\end{aligned}
$$

Since $\eta^\circ_G(x) = O(1)$ for $x \in \mathbf{R} - \sqrt{-1}y$ (see [H], Theorem 3.10), the above estimate and (32) give an analytic continuation of $\eta^\circ_{\omega_1}(\nu)$ on $-2L \le \Im(\nu) \le -2\varepsilon$.

$\eta^\circ_{\omega_2}$: In the previous sections $\varepsilon_i^\omega = \varepsilon_i^\omega(\alpha, \beta, C)$ (see 5.3 and (9)) is defined for $\alpha \ge 0$ and $\beta \ge 1$. However, under the second condition of $(B2)$ we may take $\varepsilon_i^{\omega_2} = \sigma u_{\omega_2(i)}^{-\theta}$ and easily see that all arguments in the previous sections are valid for $\varepsilon_i^{\omega_2}$, $h_i^{\omega_2}$ and $H^\circ_{\omega_2}$, especially, it follows that

$$
P_G(\nu)\eta^\circ_{\omega_2}(\nu) = \int_{\mathbf{R} - \sqrt{-1}\varepsilon} P_G(x)\eta^\circ_G(x) H^\circ_{\omega_2}(\nu, x)\,dx, \tag{34}
$$

if $\Im(\nu) \le -L$ (see Proposition 5.1). We here put $J_0 = \{i \in D_2; 1 \le \varepsilon_i^{\omega_2}\}$ and $J_n = \{i \in D_2; 2^{-n} \le \varepsilon_i^{\omega_2} < 2^{-(n-1)}\}$ $(n = 1, 2, \dots)$. Moreover, we denote by i_n the number in J_n for which $\omega_2(i_n)$ is the smallest in $\omega_2(j)(j \in J_n)$ and by $k_n(i)$ $(i \in J_n)$ the number of elements j in J_n satisfying $\omega_2(j) < \omega_2(i)$. Then for each $i \in J_n$ we see from the definition of $\delta_i^{\omega_2}$ (see 5.3 and (8)) and $(B2)$ that $u_{\omega_2(i)} \ge u_{\omega_2(i_n)} + 2\sum_{j \in J_n, \omega_2(j) < \omega_2(i)} \delta_j^{\omega_2} \ge u_{\omega_2(i_n)} + 2k_n(i)2^{-n}$ for $n \ge 0$ and $u_{\omega_2(i_n)} \ge \sigma^{1/\theta}2^{(n-1)/\theta}$ for $n \ge 1$. Therefore, by $(B2)$ and the argument used in (16)-(18) we see that if $-2L \le \Im(\nu) \le -3\varepsilon$ and $\Im(x) = -\varepsilon$,

$$
\begin{aligned}
|H^\circ_{\omega_2}(\nu, x)| &\le c \sum_{i \in D_2} \log p_i e^{(-3\varepsilon - 1/2)\log p_i} e^{(\varepsilon+1/2)u_{\omega_2(i)}} (\varepsilon_i^{\omega_2})^{1-(2M+3)} |\nu - x|^{-(2M+3)} \\
&\le c|\nu - x|^{-(2M+3)} \sum_{n=0}^{\infty} \sum_{i \in J_n} e^{-\varepsilon u_{\omega_2(i)}} (\varepsilon_i^{\omega_2})^{-2(M+1)} \\
&\le c|\nu - x|^{-(2M+3)} \Big(e^{-\varepsilon u_{\omega_2(i_0)}} \sum_{i \in J_0} e^{-2\varepsilon k_0(i)} \\
&\qquad + \sum_{n=1}^{\infty} e^{-\varepsilon \sigma^{1/\theta} 2^{(n-1)/\theta}} 2^{2n(M+1)} \sum_{i \in J_n} e^{-2\varepsilon k_n(i) 2^{-n}} \Big) \\
&\le c|\nu - x|^{-(2M+3)} \Big(\frac{1}{1 - e^{-2\varepsilon}} + \sum_{n=1}^{\infty} \frac{e^{-\varepsilon \sigma^{1/\theta} 2^{(n-1)/\theta}} 2^{2n(M+1)}}{1 - e^{-2\varepsilon 2^{-n}}} \Big) \\
&\le c|\nu - x|^{-(2M+3)}.
\end{aligned}
$$

Since $P_G(x)\eta^\circ_G(x) = O(|x|^{2M+1})$ for $x \in \mathbf{R} - \sqrt{-1}\varepsilon$ (see (23) and [H], Remark 6.8), the above estimate and (34) give an analytic continuation of $\eta^\circ_{\omega_2}(\nu)$ on $-2L \le \Im(\nu) \le -3\varepsilon$.

$\eta^\circ_{\omega_3}$: Since D_3 is finite, $\eta^\circ_{\omega_3}$ is holomorphic on the whole complex plane.

We now obtained that each $P_G(\nu)\eta^{\circ}_{\omega_\ell}(\nu)$ $(\ell = 1,2,3)$ has an analytic continuation on $-2L \leq \Im(\nu) \leq -3\varepsilon$. Therefore, $P_G(\nu)\eta^{\circ}(\nu) = \sum_{\ell=1}^{3} P_G(\nu)\eta^{\circ}_{\omega_\ell}(\nu)$ and thus, $P_G(\nu)\eta(\nu)$ have the same property (see 5.2). Since ε can be taken sufficiently small and η satisfies the functional equation (see [E], p.13), it follows that $P_G(\nu)\eta(\nu)$ is holomorphic on $0 < |\Im(\nu)| \leq 2L$. Then, noting the zeros of $P_G(\nu)$ (see (23) and (11)) and the fact that that $\zeta(s)$ has no zeros on $[0,1]$, we can finally obtain the following theorem.

Theorem 6.1. *If $SL(2,\mathbf{R})$ has a cocompact discrete subgroup Γ with $Prim_\Gamma$ satisfying the condition* (B), *then the Riemann Hypothesis holds.*

Remark 6.2. We see that $D_2 \neq \emptyset$. Actually, if $D_1 \cup D_3 = \mathbf{N}$, it follows from the above argument that $\eta^{\circ}(\nu)$ is holomorphic on $\Im(\nu) < 0$. This contradicts to the fact that $\eta(\nu)$ has a pole at $\nu = -\sqrt{-1}/2$.

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